Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(01)00086-7
J. Appl. Maths Mechs, Vol. 65, No. 5, pp. 805–816, 2001

0021-8928/01/\$—see front matter

THE SELECTION OF THE VIABILITY KERNEL FOR A DIFFERENTIAL INCLUSION†

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(*Received* 10 *September* 1999)

A controlled system and the differential inclusion corresponding to it, which function in a finite time interval and are restricted by a phase constraint in the form of a compact set in position space, are considered. A trial algorithm for the approximate construction of the viability kernel of the differential inclusion is proposed and also an algorithm for constructing the ε-viable solutions of the controlled system and the differential inclusion. © 2002 Elsevier Science Ltd. All rights reserved.

The subject matter of the paper touches on that in [1–16].‡

1. BASIC DEFINITIONS

Suppose a control system is given, the behaviour of which is described by the equation.

$$\dot{x} = f(t,x,u), \quad u \in \mathbf{P}, \ t \in \mathbf{I}, \ \mathbf{I} = [t_0, \theta], \ t_0 < \theta < \infty \tag{1.1}$$

Here, x is an m-dimensional phase vector of the system, u is a control and $\mathbf{P}$ is a compactum in the Euclidean space $\mathbf{R}^m$.

It is assumed that the following conditions are satisfied.

Condition 1. The vector-function $f(t, x, u)$ is continuous with respect to the set of variables t, x, u in the domain $\mathbf{I} \times \mathbf{R}^m \times \mathbf{P}$ and also for any bounded and closed domain $\mathbf{D} \subset \mathbf{I} \times \mathbf{R}^m$, a constant $\mathbf{L} = \mathbf{L}(\mathbf{D}) \in (0, \infty)$ exists such that

$$\| f(t,x^*,u) - f(t,x_*,u) \| \leqslant L \| x^* - x_* \|, \quad (t,x^*) \text{ and } (t,x_*) \text{ from } \mathbf{D}, \ u \in \mathbf{P}$$

Condition 2. A constant $\mu \in (0, \infty)$ exists such that

$$\| f(t,x,u) \| \leqslant \mu(1 + \|x\|), \quad (t,x,u) \in \mathbf{D} \times \mathbf{P}$$

By a permissible control $u(t)$, $t \in \mathbf{I}$, we mean any function which is Lebesgue measurable and which satisfies the inclusion $u(t) \in \mathbf{P}$, $t \in \mathbf{I}$.

We will call the absolutely continuous vector function $x[t]$, $t \in I$ which is such that $\dot{x}[t] = f(t, x[t], u[t])$ almost everywhere in $\mathbf{I}$ the solution of Eq. (1.1), that is generated by the permissible control $u(t)$, $t \in I$. We will denote the set of all $x^* \in R^m$, at which the solutions $x[t]$, $x[t_*] = x_*$ of Eq. (1.1), generated by all possible permissible controls $u(t)$, arrive at the instant t^*, by the symbol $\mathbf{Y}(t^*, t_*, x_*)$, $t_0 < t_* < t^* < \theta$. We will call $\mathbf{Y}(t^*; t_*, x_*)$ the attainability set of system (1.1) with the initial condition $x[t_*] = x_*$ corresponding to the instant t^*.

In accordance with Eq. (1.1), we set up the differential inclusion

$$\dot{x} \in \mathbf{F}(t,x), \quad t \in \mathbf{I}, \ \mathbf{F}(t,x) = \mathrm{co}\{f(t,x,u) : u \in \mathbf{P}\} \tag{1.2}$$

where co $\{\circ\}$ denotes a convex hull.

We will call the absolutely continuous vector function $x[t]$, $t \in \mathbf{I}$, which satisfies the differential inclusion (1.2) almost everywhere in $\mathbf{I}$, the solution of the differential inclusion (1.2). We will assume that $\mathbf{X}(t^*; t_*, x_*)$, $t_0 \leqslant t_* < t^* \leqslant \theta$ is the set of all $x^* \in \mathbf{R}^m$ at which all possible solutions $x[t]$, $x[t_*] = x_*$ of the differential inclusion (1.2) arrive at the instant t^*.

†*Prikl. Mat. Mekh.* Vol. 65, No. 5, pp. 831–842, 2001.

‡See also: FILIPPOVA, T. F., Viability problems for differential inclusions. Doctorate dissertation, 01.01.02, Ekaterinburg, 1992.

The equality

$$\mathbf{X}(t^*;t_*,x_*)=\mathrm{cl}\,\mathbf{Y}(t^*;t_*,x_*),\quad x_*\in\mathbf{R}^m \tag{1.3}$$

holds, where cl$\mathbf{Y}$ is the closure of the set $\mathbf{Y}$.

It is more convenient to work with closed attainability sets and we shall therefore work with the attainability set of differential inclusion (1.2).

We will assume that, together with systems (1.1) and (1.2), a closed set $\Phi\subset\mathbf{I}\times\mathbf{R}^m$ is specified which has the non-empty intersections $\Phi(t)=\{x\in\mathbf{R}^m\colon(t,x)\in\Phi\}$, $t\in\mathbf{I}$ and, moreover, $\Phi(\theta)$ is compactum in $\mathbf{R}^m$.

We shall say that the solution

$$x[t],\ t\in[t_*,\theta],\ x[t_*]=x_*,\ t_*\in\mathbf{I} \tag{1.4}$$

of differential inclusion (1.2) is viable in Φ if

$$(t,x[t])\in\Phi,\ t\in[t_*,\theta] \tag{1.5}$$

2. THE VIABILITY KERNEL

We will now consider the problem of the approximate construction of the viability kernel of differential inclusion (1.2) in the set Φ.

Definition 2.1. We will call the set of all $(t_*,x_*)\in\Phi$, which are such that solution (1.4) of differential inclusion (1.2) exists which is viable in Φ, the viability kernel Ω of differential inclusion (1.2) in the set Φ.

Taking conditions 1 and 2, which are imposed on system (1.1), into account, it can be shown that the set of all (t,x), belonging to the solutions of differential inclusion (1.2), which are viable in Φ, is contained in a certain bounded and closed set $\mathbf{D}\subset\mathbf{I}\times\mathbf{R}^m$. We shall assume, without loss of generality in the arguments, that all the points $(t,x[t])$ and all the constructions which are considered below are contained in $\mathbf{D}$.

We will use the notation

$$\begin{gathered}\omega^*(\delta)=\sup\{d(\mathbf{F}(t^*,x^*),\ F(t_*,x_*)):(t^*,x^*),(t_*,x_*)\in\mathbf{D},\ |t^*-t_*|+\|x^*-x_*\|\leqslant\delta\}\\ \omega(\delta)=\delta\omega^*((1+\mathbf{K})\delta),\ \delta>0,\ \mathbf{K}=\max\{\|f(t,x,u)\|:(t,x,u)\in\mathbf{D}\times\mathbf{P}\}<\infty\end{gathered} \tag{2.1}$$

Here, $d(\mathbf{F}^*,\mathbf{F}_*)$ is the Hausdorff distance between the sets $\mathbf{F}^*$ and $\mathbf{F}_*$.

It follows from the definition of the functions $\omega^*(\delta)$ and $\omega(\delta)$ that they decrease monotonically to zero when $\delta\to0$.

We will now specify the sequence of subdivisions $\Gamma_n=\{t_0,t_1,\dots t_{N(n)}=\theta\}$ of the interval $\mathbf{I}$ such that the diameters

$$\Delta^{(n)}=\max\{\Delta_i:0<i\leqslant N(n)-1\},\quad \Delta_i=t_{i+1}-t_i$$

of the subdivisions Γ_n tend monotonically to zero as the number n increases.

Note that the instants t_i of the subdivisions Γ_n are their own for each subdivision Γ_n. However, in order not to make the notation more complicated, we shall not explicitly reflect this dependence of the instants of time on the number n. We will assume that

$$\begin{gathered}\mathbf{X}(t^*;t_*,\mathbf{X}_*)=\bigcup_{x_*\in\mathbf{X}_*}\mathbf{X}(t^*;t_*,x_*),\ \mathbf{X}_*\subset\mathbf{R}^m\\ \mathbf{X}^{-1}(t_*;t^*,\mathbf{X}^*)=\{x_*\in\mathbf{R}^m:\mathbf{X}(t^*;t_*,x_*)\cap\mathbf{X}^*\neq\varnothing\},\quad \mathbf{X}^*\subset\mathbf{R}^m\\ \tilde{\mathbf{X}}(t^*;t_*,x_*)=x_*+(t^*-t_*)\mathbf{F}(t_*,x_*)\\ \tilde{\mathbf{X}}^{-1}(t_*,t^*,\mathbf{X}^*)=\{x_*\in\mathbf{R}^m:\tilde{\mathbf{X}}(t^*;t_*x_*)\cap\mathbf{X}^*\neq\varnothing\}\end{gathered}$$

and that $\mathbf{X}_\varepsilon$ is the ε-neighbourhood of the set $\mathbf{X}^*$. Here, $t_0\leqslant t_*<t^*\leqslant\theta$, $\mathbf{X}^*\subset\mathbf{R}^m$, $\varepsilon>0$.

We set up a sequence $\{\varepsilon_i\}$ of numbers to correspond to each subdivision Γ_n.

$$\varepsilon_i = \omega(\Delta_{i-1}) + (1 + L\Delta_{i-1})\varepsilon_{i-1}, \quad i = 1,2,\ldots,N(n); \quad \varepsilon_0 = 0$$

We also set up a sequence $\{\tilde{\Omega}^{(n)}(t_i)\}$ of sets $\tilde{\Omega}^{(n)}(t_i) \subset R^m$, $t_i \in \Gamma_n$, defined by recurrence relations, starting from the final instant $t_{N(n)} = \theta$ of the subdivision Γ_n, corresponding to each subdivision Γ_n.

Definition 2.2. We will assume that

$$\tilde{\Omega}^{(n)}(\theta) = \Phi(\theta)_{\varepsilon_{N(n)}}$$

$$\tilde{\Omega}^{(n)}(t_i) = \Phi(t_i)_{\varepsilon_i} \cap \tilde{\mathbf{X}}^{-1}(t_i; t_{i+1}, \tilde{\Omega}^{(n)}(t_{i+1})), \quad i = \mathrm{N}(n) - 1, \ \mathrm{N}(n) - 2, \ldots, 0$$

The sequence $\{\tilde{\Omega}^{(n)}(t_i)\}$ is therefore the retrogradely defined sequence of sets $\tilde{\Omega}^{(n)}(t_i)$ in $\mathbf{R}^m$. We will now determine the limit of this sequence $\{\tilde{\Omega}^{(n)}(t_i)\}$ when $\Delta^{(n)}$, the diameter of the subdivision Γ_n, tends to zero.

Definition 2.3. We will assume that Ω^0 is the set of all points $(t_*, x_*) \in \mathbf{I} \times \mathbf{R}^m$, for each of which a sequence

$$\{(\tau_n, x_n) : \tau_n = t_n(t_*), \ x_n \in \tilde{\Omega}^{(n)}(\tau_n)\} \tag{2.2}$$

$$\text{that } (t_*, x_*) = \lim_{n\to\infty} (\tau_n, x_n); \ \text{ here } \ t_n(t_*) = \min_{t_i \in \Gamma_n, t_i \geq t_*} t_i.$$

is found.

It follows from this definition that $\Omega^0 \subset \Phi$.

Note that Ω^0 is non-empty since, according to the definition, $\tilde{\Omega}^{(n)}(t_{N(n)}) = \Phi(\theta)_{\varepsilon_{N(n)}}$, and this means that the intersection $\Omega^0(\theta) = \{x: (\theta, x) \in \Omega^0\}$ is non-empty.

The following assertion holds.

Theorem 2.1. The set Ω^0 is the viability kernel of differential inclusion (1.2) in the set Φ, that is $\Omega^0 = \Omega$.

Proof. We will first prove the inclusion $\Omega^0 \subset \Omega$. We fix an arbitrary point $(t_*, x_*) \in \Omega^0$, $t_* < \theta$ and a sequence (2.2) is found such that $(t_*, x_*) = \lim_{n\to\infty} (\tau_n, x_n)$.

We now consider an arbitrary number n and the interval $[\tau_n, \theta]$ corresponding to it. It follows from the inclusion $x_n \in \tilde{\Omega}^{(n)}(\tau_n)$ that a vector function $\tilde{x}^{(n)}[t]$, which is absolutely continuous in $[\tau_n, \theta]$, exists such that

$$\begin{aligned} &\dot{\tilde{x}}^{(n)}[t] \in \mathbf{F}(t_i, \tilde{x}^{(n)}[t_i]), \quad t \in [t_i, t_{i+1}) \subset [\tau_n, \theta) \\ &\tilde{x}^{(n)}[\tau_n] = x_n, \quad \tilde{x}^{(n)}[t_i] \in \tilde{\Omega}^{(n)}(t_i), \quad \tau_n < t_i < \theta \end{aligned} \tag{2.3}$$

We now introduce functions into the treatment which are continuous extensions of the functions $\tilde{x}^{(n)}[t]$, $t \in [\tau_n, \theta]$ in the interval $[t_*, \theta]$

$$\tilde{y}^{(n)}[t] = \begin{cases} \tilde{x}^{(n)}[\tau_n], & t_* \leq t \leq \tau_n \\ \tilde{x}^{(n)}[t], & \tau_n < t \leq \theta \end{cases} \quad n = 1,2,\ldots$$

Since the sequence $\{\tilde{y}^{(n)}[t]\}$ is uniformly bounded and equipotentially continuous in $[t_*, \theta]$, a uniformly converging subsequence can be selected from it. Without loss of generality, we shall assume that the sequence $\{\tilde{y}^{(n)}[t]\}$ itself is uniformly convergent in $[t_*, \theta]$. On putting

$$x[t] = \lim_{n\to\infty} \tilde{y}^{(n)}[t], \quad t \in [t_*, \theta]$$

we obtain

$$\begin{aligned} &x[t_*] = \lim_{n\to\infty} \tilde{y}^{(n)}[t_*] = \lim_{n\to\infty} \tilde{x}^{(n)}[\tau_n] = \lim_{n\to\infty} x_n = x_* \\ &x[t] = \lim_{n\to\infty} \tilde{y}^{(n)}[t] = \lim_{n\to\infty} \tilde{x}^{(n)}[t], \quad t \in (t_*, \theta] \end{aligned} \tag{2.4}$$

If follows from condition (2.2) and (2.3) and from relations (2.4) that the vector function $x[t]$, $t \in [t_*, \theta]$ satisfies the differential inclusion

$$\dot{x} \in \mathbf{F}(t, x) \text{ almost everywhere in } [t_*, \theta] \tag{2.5}$$

and the inclusion

$$x[t] \in \Omega^0(t), \quad t \in [t_*, \theta] \tag{2.6}$$

Inclusion (2.5) is proved in the standard way (see, for example, [1, pp. 60, 61]).

We will now prove relation (2.6). We fix an arbitrary instant $t \in [t_*, \theta]$. The equality $x[t] = \lim_{n\to\infty} \tilde{y}^{(n)}[t]$ holds for this instant. By construction of the function $\tilde{y}^{(n)}[t]$, $t \in [t_*, \theta]$, the inclusion $\tilde{y}^{(n)}[t_n(t)] = \tilde{x}^{(n)}[t_n(t)] \in \tilde{\Omega}^{(n)}(t_n(t))$ is satisfied, and the instant $t_n(t)$ is defined above. We put $\eta_n = t_n(t)$ and $y_n = \tilde{x}^{(n)}[t_n(t)]$. Then,

$$\|(t, x[t]) - (\eta_n, y_n)\| \le \|(t, x[t]) - (t, \tilde{y}^{(n)}[t])\| +$$
$$+ \|(t, \tilde{y}^{(n)}[t]) - (t_n(t), \tilde{y}^{(n)}[t_n(t)])\| \le \|x[t] - \tilde{y}^{(n)}[t]\| + (1 + \mathbf{K})\Delta^{(n)}$$

On taking account of this equality and the limiting relations

$$x[t] = \lim_{n\to\infty} \tilde{y}^{(n)}[t], \quad \lim_{n\to\infty} \Delta^{(n)} = 0$$

we obtain that

$$(t, x[t]) = \lim_{n\to\infty} (\eta_n, y_n), \quad \eta_n = t_n(t), \quad y_n \in \tilde{\Omega}^{(n)}(\eta_n)$$

Inclusion (2.6) is thereby proved.

Inclusion (2.6) means that $(t, x[t]) \in \Omega^0$, $t \in [t_*, \theta]$. Then, on taking account of the inclusion $\Phi^0 \subset \Phi$, we obtain $(t, x[t]) \in \Phi$, $t \in [t_*, \theta]$.

We have thus shown that a solution (1.4) of differential inclusion (1.2), which is viable in Φ, can be found for any point $(t_*, x_*) \in \Omega^0$, $t_* < \theta$. It is also obvious that any point $(t_*, x_*) \in \Omega^0$, $t_* = \theta$ satisfies the inclusion $(t_*, x_*) \in \Phi$. At the same time, it has been shown that $\Omega^0 \subset \Omega$.

We will now prove the inverse inclusion $\Omega \subset \Omega^0$.

Consider a subdivision Γ_n of the interval $\mathbf{I}$ and all of those intersections $\Omega(t_i)$, $t_i \in \Gamma_n$ of the set Ω which are non-empty. The notation $\mathrm{T}_n = \{t_i \in \Gamma_n : \Omega(t_i) \neq \emptyset\}$ is used. It is clear that the set T_n possesses the following property: if $t_i \in \mathrm{T}_n$, then $t_{i+1} \in \mathrm{T}_n$.

The inclusions

$$\Omega(t_i) \subset \Phi(t_i) \cap \mathbf{X}^{-1}(t_i; t_{i+1}, \Omega(t_{i+1})), \quad t_i \in \mathrm{T}_n \tag{2.7}$$

hold.

Actually, suppose $x^{(i)} \in \Omega(t_i)$. Then, a solution $x[t]$, $t \in [t_i, \theta]$, $x[t_i] = x^{(i)}$ of differential inclusion (1.2) is found which is viable in Φ. Since the solution $x[t]$, $t \in [t_{i+1}, \theta]$ is viable in Φ, then $x[t_{i+1}] \in \Omega(t_{i+1})$. It follows from this that

$$\mathbf{X}(t_{i+1}; t_i, x[t_i]) \cap \Omega(t_{i+1}) \neq \emptyset$$

and this means that

$$\Omega(t_i) \subset \mathbf{X}^{-1}(t_i; t_{i+1}, \Omega(t_{i+1}))$$

On also taking account of the inclusion $\Omega(t_i) \subset \Phi(t_i)$, we obtain relations (2.7).

We now choose an arbitrary instant $t_i \in \mathrm{T}_n$, $t_i < \theta$ and consider the sets $\Omega(t_i)$ and $\Omega(t_{i+1})_{\omega(\Delta_i)}$; the numbers $\omega(\Delta_i)$ are defined above.

The inclusion

$$\Omega(t_i) \subset \tilde{X}^{-1}(t_i; t_{i+1}, \Omega(t_{i+1})_{\omega(\Delta_i)}), \quad t_i \in \mathrm{T}_n \tag{2.8}$$

holds.

We will now prove this. Suppose $x[t_i] \in \Omega(t_i)$. Each point $x[t_{i+1}] \in \mathbf{X}(t_{i+1}; t_i, x[t_i])$ is a finite value of a certain solution $x[t]$, $t \in [t_i, t_{i+1}]$ of the differential inclusion $\dot{x} \in \mathbf{F}(t, x)$, $t \in [t_i, t_{i+1}]$ with an initial value $x[t_i]$. The equality

$$x[t_{i+1}] = x[t_i] + \mathbf{I}_i, \quad \mathbf{I}_i = \int_{t_i}^{t_{i+1}} f[t]dt, \quad f[t] \in \mathbf{F}(t, x[t]), \quad t \in [t_i, t_{i+1}]$$

holds.

On taking account of the definition of the function $\omega^*(\Delta)$, we obtain

$$d(\mathbf{F}(t, x[t]),\ \mathbf{F}(t_i, x[t_i])) \leq \omega^*(|t - t_i| + \|x[t] - x[t_i]\|) \leq \omega^*((1 + \mathbf{K})\Delta_i), \quad t \in [t_i, t_{i+1}] \tag{2.9}$$

This means that the inclusion

$$f[t] \in \mathbf{F}(t_i, x[t_i])_{\omega^*((1+\mathbf{K})\Delta_i)}, \quad t \in [t_i, t_{i+1}]$$

holds, from which the inclusion

$$\frac{1}{\Delta_i}\mathbf{I}_i \in \mathbf{F}(t_i, x[t_i])_{\omega^*((1+\mathbf{K})\Delta_i)} \tag{2.10}$$

follows.

The inclusion

$$x[t_{i+1}] \in x[t_i] + \Delta_i \mathbf{F}(t_i, x[t_i])_{\omega^*((1+K)\Delta_i)} = \tilde{\mathbf{X}}(t_{i+1}; t_i, x[t_i]]_{\omega(\Delta_i)} \tag{2.11}$$

follows from (2.10).

On taking account of the fact that relation (2.11) was obtained for an arbitrary point $x[t_{i+1}] \in X(t_{i+1}); t_i, x[t_i]$, we conclude that

$$\mathbf{X}(t_{i+1}; t_i, x[t_i]) \subset \tilde{\mathbf{X}}(t_{i+1}; t_i, x[t_i])_{\omega(\Delta_i)}$$

It follows from the inclusion $x[t_i] \in \Omega(t_i)$ that

$$\Omega(t_{i+1}) \cap \mathbf{X}(t_{i+1}; t_i, x[t_i]) \neq \varnothing$$

and this means that

$$\Omega(t_{i+1})_{\omega(\Delta_i)} \cap \tilde{\mathbf{X}}(t_{i+1}; t_i, x[t_i]) \neq \varnothing \tag{2.12}$$

Since the instant $t_i \in \mathrm{T}_n$ and the point $x[t_i] \in \Omega(t_i)$ were chosen in an arbitrary manner, the inclusion (2.8) follows from (2.12).

We now define a system $\{\hat{\Omega}^{(n)}(t_i) : t_i \in \mathrm{T}_n\}$ of sets $\hat{\Omega}^{(n)}(t_i)$ by the equalities $\hat{\Omega}^{(n)}(t_i) = \Omega(t_i)_{\varepsilon_i}$ (the numbers ε_i are defined above at the beginning of section 2). According to the definition of the sets $\{\hat{\Omega}^{(n)}(t_i), t_i \in \mathrm{T}_n\}$, the inclusions

$$\Omega(t_i) \subset \hat{\Omega}^{(n)}(t_i), \quad t_i \in \mathrm{T}_n$$

are satisfied.

The inclusions

$$\hat{\Omega}^{(n)}(t_i) \subset \tilde{X}^{-1}(t_i; t_{i+1}, \hat{\Omega}^{(n)}(t_{i+1})), \quad t_i \in \mathrm{T}_n \tag{2.13}$$

hold.

We will now prove this. Suppose $x[t_i] \in \hat{\Omega}^{(n)}(t_i)$ and $x^*[t_i]$ is the closest point in $\Omega(t_i)$ to the point $x[t_i]$. The inequality $\|x[t_i] - x^*[t_i]\| \leq \varepsilon_i$ holds.

The relation

$$\Omega(t_{i+1})_{\omega(\Delta_i)} \cap \tilde{X}(t_{i+1}; t, x^*[t_i]) \neq \varnothing \tag{2.14}$$

follows from the inclusions $x^*[t_i] \in \Omega(t)$ and (2.8).
A point

$$x^*[t_{i+1}] = x^*[t_i] + \Delta_i f^*[t_i], \quad f^*[t_i] \in \mathbf{F}(t_i, x^*[t_i]) \tag{2.15}$$

then exists which is contained in $\Omega(t_{i+1})_{\omega(\Delta_i)}$.
Taking account of the inequality

$$d(\mathbf{F}(t_i, x[t_i]), \mathbf{F}(t_i, x^*[t_i])) \leqslant \mathbf{L} \, \|x[t_i] - x^*[t_i]\|$$

we choose a vector $f[t_i] \in \mathbf{F}(t_i, x[t_i])$ which satisfies the inequality

$$\|f[t_i] - f^*[t_i]\| \leqslant \mathbf{L} \| x[t_i] - x^*[t_i]\| \leqslant \mathbf{L}\varepsilon_i$$

It is then found that the point $x[t_{i+1}] = x[t_i] + \Delta_i f[t_i]$ is spaced a distance from the point (2.15) not greater than by the amount

$$\|x[t_i] - x^*[t_i]\| + \Delta_i \|f[t_i] - f^*[t_i]\| \leqslant (1 + \mathbf{L}\Delta_i)\varepsilon_i$$

This means that

$$x[t_{i+1}] \in \hat{\Omega}_n(t_{i+1})$$

Hence, it has been shown that the relation

$$\tilde{\mathbf{X}}(t_{i+1}; t_i, x[t_i]) \cap \hat{\Omega}^{(n)}(t_{i+1}) \neq \varnothing$$

holds for any $t_i \in T_n$, $x[t_i] \in \hat{\Omega}^{(n)}(t_i)$, and inclusion (2.13) follows from this.
The inclusions

$$\hat{\Omega}^{(n)}(t_i) \subset \tilde{\Omega}^{(n)}(t_i), \quad t_i \in \mathbf{T}_n \tag{2.16}$$

hold.

We will now prove (2.16) by mathematical induction. In fact, the relations

$$\hat{\Omega}^{(n)}(t_i) = \Omega(t_i)_{\varepsilon_i} \subset \Phi(t_i)_{\varepsilon_i}, \quad t_i \in \mathbf{T}_n \tag{2.17}$$

$$\hat{\Omega}^{(n)}(t_{N(n)}) = \Omega(t_{N(n)})_{\varepsilon_{N(n)}} = \Phi(t_{N(n)})_{\varepsilon_{N(n)}} = \tilde{\Omega}^{(n)}(t_{N(n)}) \tag{2.18}$$

are satisfied. Consequently, $\hat{\Omega}^{(n)}(t_i) \subset \tilde{\Omega}^{(n)}(t_i)$ is satisfied for $i = N(n)$.
We will prove that this inclusion is satisfied for all remaining i for which $t_i \in \mathrm{T}_n$.
To do this, we assume that $t_i \in \mathrm{T}_n$ and that the inclusion

$$\hat{\Omega}^{(n)}(t_{i+1}) \subset \tilde{\Omega}^{(n)}(t_{i+1}) \tag{2.19}$$

holds for the instant t_{i+1}.
We will now prove that $\hat{\Omega}^{(n)}(t_i) \subset \tilde{\Omega}^{(n)}(t_i)$. Actually, it follows from (2.13) and (2.17) that

$$\hat{\Omega}^{(n)}(t_i) \subset \Phi(t_i)_{\varepsilon_i} \cap \tilde{X}^{-1}(t_i; t_{i+1}, \hat{\Omega}^{(n)}(t_{i+1}))$$

and it follows from (2.19) that $\Phi(t_i)_{\varepsilon_i} \cap \tilde{X}^{-1}(t_i; t_{i+1}, \hat{\Omega}^{(n)}(t_{i+1})) \subset \Phi(t_i)_{\varepsilon_i} \cap \tilde{X}^{-1}(t_i; t_{i+1}, \tilde{\Omega}^{(n)}(t_{i+1}))$. From this we obtain $\hat{\Omega}^{(n)}(t_i) \subset \tilde{\Omega}^{(n)}(t_i)$. At the same time, relations (2.16) have been proved.
We will now use relation (2.16) to prove the inclusion $\Omega \subset \Omega^0$.

In the case when $t_* = \theta$, the equalities

$$\Omega(t_*) = \Phi(\theta), \quad \Omega^0(t_*) = \Phi(\theta)$$

are satisfied and this means that $\Omega(t_*) = \Omega^0(t_*)$.

Suppose $t_* < \theta$. We choose an arbitrary point $(t_*, x_*) \in \Omega$. The inequalities

$$t_* < t_n(t_*) \leqslant t_* + \Delta^{(n)}, \quad n = 1, 2, \ldots$$

hold.

Since $(t_*, x_*) \in \Omega$, a solution $x[t]$, $t \in [t_*, \theta]$, $x[t_*] = x_*$ of differential inclusion (1.2) exists which is viable in Φ. Any "piece" $x[t]$, $t \in [t^*, \theta]$, $t_* \leqslant t^* \leqslant \theta$ of this solution is a solution of differential inclusion (1.2) which is viable in Φ. The inclusion

$$x[t_n(t_*)] \in \Omega(t_n(t_*)) \subset \hat{\Omega}^{(n)}(t_n(t_*)) \subset \tilde{\Omega}^{(n)}(t_n(t_*))$$

follows from this.

This means that a point $x[t_n(t_*)]$ is found for each n such that

$$x[t_n(t_*)] \in \tilde{\Omega}(t_n(t_*)), \quad \| x[t_n(t_*)] - x_* \| \leqslant \mathbf{K}(t_n(t_*) - t_*)$$

On taking account of the equality

$$\lim_{n \to \infty} (t_n(t_*) - t_*) = 0$$

we obtain that the sequence $\{(t_n(t_*), x[t_n(t_*)])\}$ satisfies the relation

$$\lim_{n \to \infty} (t_n(t_*), x[t_n(t_*)]) = (t_*, x_*)$$

and this means that $(t_*, x_*) \in \Omega^0$.

It has been shown that $\Omega(t_*)$, $\subset \Omega^0(t_*)$, $t_* < \theta$.

The inclusion $\Omega \subset \Omega^0$ follows from the relations $\Omega(\theta) \subset \Omega^0(\theta)$ and $\Omega(t_*) \subset \Omega^0(t_*)$, $t_* < \theta$. From the inclusions $\Omega^0 \subset \Omega$, $\Omega \subset \Omega^0$, it follows that $\Omega = \Omega^0$. Theorem 2.1. is proved.

3. THE CONSTRUCTION OF E-VIABLE SOLUTIONS

We will now propose a procedure for constructing ε-viable solutions of control system (1.1) and differential inclusion (1.2). This procedure is a well-known control procedure with a guide [4], which has been adapted for solving viability problems. We will assume that the following condition is satisfied with respect to the set Φ.

Condition 3. The inequality

$$\sup_{t_*, t^* \in I, |t^* - t_*| \leqslant \delta} d(\Phi(t^*), \Phi(t_*)) \leqslant \chi(\delta), \quad \delta > 0$$

is satisfied, where the function $\chi(\delta)$ satisfies the limiting relation $\lim_{\delta \downarrow 0} \chi(\delta) = 0$. Here, $d(\Phi(t^*), \Phi(t_*))$ is the Hausdorff distance between the intersections $\Phi(t_*)$ and $\Phi(t^*)$ of the set Φ.

We will now describe a control procedure with a guide. Suppose Γ_n is a certain subdivision from the sequence of subdivisions of the interval $\mathbf{I}$, which has been defined in section 2. We will assume that we have already calculated all of the sets $\tilde{\Omega}^{(n)}(t_i)$, $t_i \in T_n$ defined by the relations

$$\tilde{\Omega}^{(n)}(\theta) = \Phi(\theta)_{\varepsilon_{N(n)}}, \quad \tilde{\Omega}^{(n)}(t_i) = \Phi(t_i)_{\varepsilon_i} \cap \tilde{X}^{-1}(t_i; t_{i+1}, \tilde{\Omega}^{(n)}(t_{i+1})), \quad t_i, t_{i+1} \in \mathrm{T}_n$$

Suppose t_k is the least instant in the set T_n. We consider an arbitrary point $x_* \in \tilde{\Omega}^{(n)}(t_k)$ and put $x[t_k] = z[t_k] = x_*$. A vector $f[t_k] \in \mathbf{F}(t_k, z[t_k])$ is found such that

$$z[t_{k+1}] = z[t_k] + \Delta_k f[t_k] \in \tilde{\Omega}^{(n)}(t_{k+1}) \tag{3.1}$$

Since $z[t_{k+1}] \in \tilde{\Omega}^{(n)}(t_{k+1})$, a vector $f[t_{k+1}] \in F(t_{k+1}, z[t_{k+1}])$ is found such that relation (3.1) is satisfied when k is replaced by $k + 1$.

We assume that the point $z[t_{k+p}] \in \tilde{\Omega}^{(n)}(t_{k+p})$ has been calculated at a certain instant $t_{k+p} \in \mathrm{T}_n$. We then determine the following point $z[t_{k+p+1}]$ using relation (3.1) with k replaced by $k + p$, where $f[t_{k+p}]$ is a certain vector from $\mathbf{F}(t_{k+p}, z[t_{k+p}])$. Thus, by continuing to construct the points $z[t_k], z[t_{k+1}], \ldots, z[t_{k+p}], z[t_{k+p+1}], \ldots$, successively, we finally construct the last point such that relation (3.1) is satisfied when $k = \mathbf{N}(n) - 1$, where

$$z[t_{\mathbf{N}(n)-1}] \in \tilde{\mathbf{\Omega}}^{(n)}(t_{N(n)-1}), f[t_{\mathbf{N}(n)-1}] \in \mathbf{F}(t_{\mathbf{N}(n)-1}, z[t_{\mathbf{N}(n)-1}])$$

The sequence $\{z[t_i]\}$ $(i = k, k + 1, \ldots, N(n))$ satisfies the relations $z[t_i] \in \Phi(t_i)_{\varepsilon_i}$ and it can be represented as a discretely defined motion of a guide for control system (1.1) in the interval **I**. We supplement the discretely defined motion of the guide in the whole of the interval $[t_*, \theta]$ by putting

$$z[t] = z[t_i] + (t - t_i)f[t_i], \quad t \in [t_i, t_{i+1}), \quad i = k, k + 1, \ldots, \mathbf{N}(n) - 1$$

Using the motion of the guide $z[t]$, $t \in [t_k, \theta]$, we determine the control

$$u(t) = u_i, \quad t \in [t_i, t_{i+1}), \quad u_i \in \mathbf{P}, i = k, k + 1, \ldots, \mathbf{N}(n) - 1$$

which generates the ε-viable solution of system (1.1) We determine the sequence $u_k, u_{k+1}, \ldots, u_{\mathbf{N}(n)-1}$ as follows. We consider an interval $[t_k, t_{k+1})$ of the subdivisions Γ_n and arbitrarily choose a vector $u_k \in P$. The solution $x[t]$, $t \in [t_k, t_{k+1}]$ of system (1.1), generated by the control $u(t) = u_k$, $t \in [t_k, t_{k+1})$ and with an initial condition $x[t_k] = x_*$, satisfies the relation

$$x[t] = x_* + \Phi(t_k, t), \quad \Phi(t_k, t) = \int_{t_*}^{t} f(\tau, x[\tau], u(\tau))d\tau, \; t \in [t_k, t_{k+1}]$$

We now consider the interval $[t_{k+1}, t_{k+2})$ and the vector $s[t_{k+1}] = z[t_{k+1}] - x[t_{k+1}]$.

If $s[t_{k+1}] = 0$, we choose an arbitrary vector $u_{k+1} \in P$

If $s[t_{k+1}] \neq 0$, we choose a vector $u_{k+1} \in P$ from the condition

$$s[t_{k+1}]' f(t_{k+1}, x[t_{k+1}], u_{k+1}) = \max_{u \in \mathbf{P}} s[t_{k+1}]' f(t_{k+1}, x[t_{k+1}], u)$$

where $s'f$ denotes the scalar product of the vectors s and f.

The solution $x[t]$, $t \in [t_{k+1}, t_{k+2}]$ of system (1.1), which is generated by the control $u(t) = u_{k+1}$, $t \in [t_{k+1}, t_{k+2})$, satisfies the relation

$$x[t] = x[t_{k+1}] + \Phi(t_{k+1}, t), \; t \in [t_{k+1}, t_{k+2}]$$

Next, we calculate the vector $s[t_{k+2}] = z[t_{k+2}] - x[t_{k+2}]$ and, as before, we choose a vector $u_{k+2} \in \mathbf{P}$. Thus, by continuing to construct the vectors $s[t_i] = z[t_i] - x[t_i]$ and $u_i (i = k, k + 1, \ldots, \mathbf{N}(n) - 1)$ successively, we also determine in parallel the solution of system (1.1) which is generated by the piecewise-constant control

$$u(t) = u_i, \quad t \in [t_i, t_{i+1}), \quad i = k, k + 1, \ldots, \mathbf{N}(n) - 1$$

We have

$$x[t] = x[t_i] + \Phi(t_i, t), \quad t \in [t_i, t_{i+1}], \quad i = k, k + 1, \ldots, \mathbf{N}(n) - 1$$

We will now derive an upper limit of the square of the magnitude of the deviation of the solution $x[t]$ from the set $\Phi(t)$, $t \in [t_k, \theta]$. This deviation will be denoted by the symbol

$$d(x[t], \Phi(t)) = \min_{w \in \Phi(t)} \| x[t] - w \|$$

To do this, we first derive an upper limit of the quantity $\|s[t_i]\|^2$ in terms of the initial magnitude of

$\|s[t_k]\|^2$ (the notation $s[t_k] = z[t_k] - x[t_k] = 0$ is used here).

The equality

$$s[t_{i+1}] = s[t_i] + \int_{t_i}^{t_{i+1}} f[t_i]dt - \int_{t_i}^{t_{i+1}} f(t, x[t], u(t))dt \tag{3.2}$$

$$f[t_i] \in \mathbf{F}(t_i, z(t_i]),\ u_i \in \mathbf{U}_{s[t_i]}(t_i, x[t_i]),\ \mathbf{U}_{s[t_i]}(t_i, x[t_i]) = \{u_* \in \mathbf{P} : s[t_i]' f(t_i, x[t_i], u_*) = \\ = \max_{u \in \mathbf{P}} s[t_i]' f(t_i, x[t_i], u)\},\ s[t_i] = z[t_i] - x[t_i]$$

$$\| s[t_{i+1}]\|^2 = \| s[t_i]\|^2 + 2\int_{t_i}^{t_{i+1}} s[t_i]'(f[t_i] - f(t, x[t], u_i))dt + \gamma(t_i, t_{i+1})$$

$$\gamma(t_i, t_{i+1}) = \left(\int_{t_i}^{t_{i+1}} (f[t_i] - f(t, x[t], u_i))dt \right)^2$$

holds for each i, $k \leqslant i \leqslant N(n) - 1$.

Using arguments which are standard in the theory of differential games (see [4]), we obtain

$$s[t_i]'(f[t_i] - f(t, x[t], u_i)) \leqslant \mathbf{L} \| s[t_i]\|^2 + \gamma\omega^*((1 + \mathbf{K})\Delta_i)$$

$$\gamma = \gamma(\mathbf{D}) = \max\{\|(t, w_*) - (t, w^*)\| : (t, w_*), (t, w^*) \in \mathbf{D}\} < \infty$$

whence we find

$$2\int_{t_i}^{t_{i+1}} s[t_i]'(f[t_i] - f(t, x[t], u_i))dt \leqslant 2\mathbf{L} \| s[t_i]\|^2 \Delta_i + \gamma\omega^*((1 + \mathbf{K})\Delta_i)\Delta_i \tag{3.3}$$

The limit

$$\gamma(t_i, t_{i+1}) \leqslant 4\mathbf{K}^2 \Delta_i^2 \tag{3.4}$$

holds for the quantity $\gamma(t_i, t_{i+1})$.

Taking (3.3) and (3.4) into account, we obtain

$$\| s[t_{i+1}]\|^2 \leqslant \| s[t_i]\|^2 + 2\mathbf{L}\Delta_i \| s[t_i]\|^2 + \gamma\omega^*((1 + \mathbf{K})\Delta_i)\Delta_i + 4\mathbf{K}^2\Delta_i^2, \quad i = k, k+1, \dots, N(n) - 1$$

and, from this, the limit

$$\| s[t_{i+1}]\|^2 \leqslant e^{2L\Delta_i} \| s[t_i]\|^2 + \Delta_i \varphi(\Delta_i),\ i = k, k+1, \dots, N(n) - 1 \tag{3.5}$$

follows, where

$$\varphi(\Delta) = \gamma\omega^*((1 + \mathbf{K})\Delta) + 4\mathbf{K}^2\Delta^2,\quad \Delta > 0$$

is a quantity which tends to zero as $\Delta \to 0$ and is independent of the choice of the points $(t_i, x[t_i])$ and $(t_i, z[t_i])$.

The limit

$$\| s[t_i]\|^2 \leqslant e^{2L(\theta - t_0)}(\| s[t_k]\|^2 + (\theta - t_0)\varphi(\Delta^{(n)})),\ i = k+1, k+2, \dots, N(n)$$

is obtained from the limit (3.5) by successive substitution of its upper limit instead of the quantity $\|s[t_i]\|^2$. Since $\|s[t_k]\| = 0$, we then obtain

$$\| s[t_i]\|^2 \leqslant e^{2L(\theta - t_0)}(\theta - t_0)\varphi(\Delta^{(n)}),\ i = k+1, k+2, \dots, N(n) \tag{3.6}$$

Further, $d(\Phi(t_i), \Phi(t)) \leqslant \chi(\Delta_i)$ according to Condition 3. The inequality

$$\|z[t] - z[t_i]\| \leqslant (t - t_i)\|f[\tau_i]\| \leqslant \mathbf{K}\Delta_i, \quad t \in [t_i, t_{i+1}]$$

is also satisfied, and it follows from this that

$$d(z[t], \Phi(t)) \leqslant \| z[t] - z[t_i] \| + d(z[t_i], \Phi(t_i)) + d(\Phi(t_i), \Phi(t)) \leqslant \mathbf{K}\Delta_i + \varepsilon_i + \chi(\Delta_i)$$

and this means that

$$d(z[t], \Phi(t)) \leqslant \mathbf{K}\Delta^{(n)} + \varepsilon_{\mathbf{N}(n)} + \chi(\Delta^{(n)}), \quad t \in [t_i, t_{i+1}], \quad i = k, k+1, \ldots, \mathbf{N}(n) - 1 \tag{3.7}$$

Hence, inequality (3.7) holds for any point $z[t]$, $t \in [t_k, \theta]$.
Taking account of the limit

$$\|x[t] - x[t_i]\| \leqslant \mathbf{K}\Delta_i \leqslant \mathbf{K}\Delta^{(n)}, \quad \|z[t] - z[t_i]\| \leqslant \mathbf{K}\Delta_i \leqslant \mathbf{K}\Delta^{(n)}$$

when $t \in [t_i, t_{i+1}]$, $i = k, k + 1, \ldots \mathbf{N}(n) - 1$ and, also, limit (3.6), we obtain

$$\begin{aligned} &\| x[t] - z[t] \| \leqslant \| x[t] - x[t_i] \| + \| x[t_i] - z[t_i] \| + \| z[t_i] - z[t] \| \leqslant 2\mathbf{K}\Delta^{(n)} + \\ &+ e^{L(\theta - t_0)}(\theta - t_0)^{1/2}\varphi(\Delta^{(n)})^{1/2} \end{aligned}$$

In turn, we find from this

$$\begin{aligned} &d(x[t], \Phi(t)) \leqslant \|x[t] - z[t]\| + d(z[t], \Phi(t)) \leqslant \\ &\leqslant 2\mathbf{K}\Delta^{(n)} + e^{L(\theta - t_0)}(\theta - t_0)^{1/2}\varphi(\Delta^{(n)})^{1/2} + \mathbf{K}\Delta^{(n)} + \varepsilon_{N(n)} + \chi(\Delta^{(n)}) \end{aligned} \tag{3.8}$$

The limit

$$\varepsilon_{\mathbf{N}(n)} \leqslant e^{L(\theta - t_0)}(\theta - t_0)\omega^*((1 + \mathbf{K})\Delta^{(n)}) \tag{3.9}$$

holds for the quantity $\varepsilon_{\mathbf{N}(n)}$ on the right-hand side of inequality (3.8).
Taking inequality (3.9) into account, from (3.8) we obtain the limit

$$\begin{aligned} &d(x[t], \Phi(t)) \leqslant 3\mathbf{K}\Delta^{(n)} + e^{L(\theta - t_0)}(\theta - t_0)^{1/2}\varphi(\Delta^{(n)})^{1/2} + \\ &+ e^{L(\theta - t_0)}(\theta - t_0)\omega^*((1 + \mathbf{K})\Delta^{(n)}) + \chi(\Delta^{(n)}) \end{aligned} \tag{3.10}$$

Since the functions $\varphi(\Delta)$, $\omega^*((1 + K)\Delta)$, $\chi(\Delta)$ satisfy the limiting relations

$$\lim_{\Delta \downarrow 0} \varphi(\Delta) = 0, \quad \lim_{\Delta \downarrow 0} \omega^*((1 + \mathbf{K})\Delta) = 0, \quad \lim_{\Delta \downarrow 0} \chi(\Delta) = 0,$$

and the sequence of subdivisions $\{\Gamma_n\}$ of the interval **I** is such that $\lim_{n \to \infty} \Delta^{(n)} = 0$, the right-hand side of inequality (3.10) tends to zero as $n \to \infty$. It follows from this that the following assertion holds.

Theorem 3.1. Suppose control system (1.1) satisfies Conditions 1 and 2, and the set – constraint Φ satisfies Condition 3. Then, for any $\varepsilon > 0$, a number $n_* = n_*(\varepsilon)$ is found such that for every $n \geqslant n_*$ and any point $x_* \in \tilde{\Omega}^{(n)}(t_k)$($t_k$ is the least instant in the set T_n), a permissible control is found that generates a solution $x[t]$, $t \in [t_k, \theta]$ of system (1.1) which satisfies the inequality $d(x[t], \Phi(t)) \leqslant \varepsilon$, $t \in [t_k, \theta]$.

Remark 1. The construction of an ε-viable solution $x[t]$ is possible for the approach described here only if the functions $\varphi(\delta)$, $\omega^*((1 + K)\delta)$ and $\chi(\delta)$ of the variable $\delta > 0$ are known. In the case of this condition, on fixing $\varepsilon \in (0, \infty)$, we find a number $n_* = n_*(\varepsilon)$ such that, when $n = n_*$, the following inequalities, for example, hold

$$\begin{aligned} &3K\Delta^{(n)} \leqslant \varepsilon/4, \quad e^{L(\theta - t_0)}(\theta - t_0)^{1/2}\varphi(\Delta^{(n)})^{1/2} \leqslant \varepsilon/4 \\ &e^{L(\theta - t_0)}(\theta - t_0)^{1/2}\omega^*((1 + \mathbf{K})\Delta^{(n)}) \leqslant \varepsilon/4, \quad \chi(\Delta^{(n)}) \leqslant \varepsilon/4 \end{aligned} \tag{3.11}$$

Such a choice of the number $n = n_*$ ensures the existence of a solution $x[t]$ of control system (1.1) ($x[t_k] = x_* \in \tilde{\Omega}^{(n)}(t_k)$), which satisfies the inequality $d(x[t], \Phi(t)) \leq \varepsilon$, $t \in [t_k, \theta]$, that is, which is ε-viable in Φ.

By virtue of the monotonic decrease of the diameters $\Delta^{(n)}$ to zero when $n \to \infty$, we also obtain that, for any $n \geq n_*$ and for any point $x_* \in \tilde{\Omega}^{(n)}(t_k)$, a permissible control $u(t)$, $t \in [t_k, \theta]$ is also found which generates an ε-viable solution of control system (1.1). This control $u(t)$, $t \in [t_k, \theta]$ can be formulated as a piecewise-constant control for system (1.1) in a control procedure with a guide corresponding to the subdivision Γ_n.

Remark 2. A permissible piecewise-constant control $u(t)$, $t \in [t_i, \theta)]$, that generates a solution of system (1.1) which is ε-viable in Φ, is found in the case of the numbers n, to which the diameters $\Delta^{(n)}$ that satisfy inequalities (3.11) correspond, not only for the points $x_* \in \tilde{\Omega}^{(n)}(t_k)$ but also for every point $x_* \in \tilde{\Omega}^{(n)}(t_i)$, $t_i \in \mathrm{T}_n$.

Remark 3. In the case when control system (1.1) is autonomous, that is, the vector function $f(t,x)$ does not explicitly depend on t, the function $\omega^*(\delta)$ (see the beginning of section 2) takes a form which differs from (2.1) in that $d(\mathbf{F}(t^*, x^*), \mathbf{F}(t_*, x_*))$ is replaced by $d(\mathbf{F}(x^*), \mathbf{F}(x_*))$ and, moreover, $d(\mathbf{F}(x^*), \mathbf{F}(x_*)) \leq L\|x^* - x_*\|$.

In this case, limit (2.9) for the change of the set $\mathbf{F}(t, x)$ along the solution $x[t]$, $t \in [t_i, t_{i+1}]$ can be replaced by the limit

$$d(\mathbf{F}(x[t]), \mathbf{F}(x[t_i])) \leq \mathbf{L}\,\|x[t] - x[t_i]\| \leq \mathbf{LK}\Delta_i, \quad t \in [t_i, t_{i+1}]$$

Hence, in the case when control system (1.1) is autonomous, the function $\omega^*((1 + K)\delta$, $\delta > 0$ in section 2 can be replaced by the function $\omega^*(\delta) = \mathbf{LK}\delta$, $\delta > 0$ and, if it is also permitted that the function $\chi(\delta)$, $\delta > 0$ has the form $\chi(\delta) = \alpha\delta^\beta$, $\delta > 0$, where α and β are certain positive constants, then limits (3.11) take the form

$$3\mathbf{K}\Delta^{(n)} \leq \varepsilon/4, \; e^{\mathbf{L}(\theta - t_0)}(\theta - t_0)^{1/2}(\gamma L\mathbf{K}\Delta^{(n)} + 4\mathbf{K}^2\Delta^{(n)^2})^{1/2} \leq \varepsilon/4$$

$$e^{\mathbf{L}(\theta - t_0)}(\theta - t_0)^{1/2}\mathbf{LK}\Delta^{(n)} \leq \varepsilon/4, \; \alpha\Delta^{(n)^\beta} \leq \varepsilon/4$$

Then, the number n_*, about which we have spoken in Theorem (3.1), can be determined as simultaneously satisfying the four equalities

$$\Delta^{(n_*)} \leq \frac{\varepsilon}{12\mathbf{K}}, \; \Delta^{(n_*)} \leq \frac{\varepsilon^2}{16e^{2\mathbf{L}(\theta - t_0)}(\gamma\mathbf{LK} + 4\mathbf{K}^2)}$$

$$\Delta^{(n_*)} \leq \frac{\varepsilon}{4e^{L(\theta - t_0)}(\theta - t_0)^{1/2}\mathbf{LK}}, \quad \Delta^{(n_*)} \leq \left(\frac{\varepsilon}{4\alpha}\right)^{1/\beta}$$

Since the diameters $\Delta^{(n)}$ decrease monotonically to zero as $n \to \infty$, a permissible control $u(t)$, $t \in [t_k, \theta]$ which generates an ε-viable solution $x[t]$ of control system (1.1) is found for every $n \geq n_*$ and any point $x_* \in \tilde{\Omega}^{(n)}(t_k)$.

This research was supported financially by the Russian Foundation for Basic Research (99-01-00146, 96-15-96245).

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Translated by E.L.S.